Journal of Global Optimization **29:** 85–95, 2004.

Some Convergence Results for Evolution Hemivariational Inequalities

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(Received 27 June 2003; accepted 12 October 2003)

Abstract. This paper is devoted to the regularization of a class of evolution hemivariational inequalities. The operator involved is taken to be non-coercive and the data are assumed to be known approximately. Under the assumption that the evolution hemivariational inequality be solvable, a weakly convergent approximation procedure is designed by means of the so-called Browder–Tikhonov regularization method.

Mathematics Subject Classifications (1991). 34G05, 47H19.

Key words: convergence, hemivariational inequalities, monotone operators, regularization, the Clarke subdifferential.

1. Introduction

Let H be a separable Hilbert space and V be a dense subspace of H carrying the structure of a separable reflexive Banach space. We assume that V compactly imbeds into H. Identifying H with its dual, we obtain $V \subseteq H \subseteq V'$ (V' is the dual space of V) which forms an evolution triple. The norm of any Banach space B is denoted by $\|\cdot\|_B$. The pairing between B and its dual space B' is denoted by $\langle \cdot,\cdot \rangle_B$. Let $0 < T < +\infty$, $I \equiv [0,T]$. For each $r \geqslant 1$, we denote by $L^r(I,B)$ the space of strongly measurable B-valued functions b: $I \to B$ such that $\int_1 \|b(t)\|_B^r \mathrm{d}t < +\infty$. Let $X = L^2(I,V), Z = L^2(I,H)$ and X', Z' the dual spaces of X, Z respectively, i.e., $X' = L^2(I,V'), Z' = L^2(I,H)$ (see Zeidler, 1990, pp. 412). We may assume without loss of generality that X and X' are locally uniformly convex (see Zeidler, 1990, pp. 420, for example). The norm convergence in X and X' is denoted by $\to$, and the weak convergence by $\rightharpoonup$.

Let A: $X \to X'$ be an operator and $G^0(\cdot,\cdot)$ stand for the Clarke's directional derivative (see below, for the definition). We shall deal with the following evolution hemivariational inequality:

*Financed partially by: NNSF of China Grant No.10171008, Education Department of China Grant No.01084, Hunan Education Department of China Grant No.01A025 and the Lady Davis Fellowship at the Technion in Israel.

Problem P. Find $u \in X$ such that $u(0)=u(T)$ and

$$\langle \dot{u}, v\rangle_X + \langle Au, v\rangle_X + G^0(u,v) \geqslant \langle f, v\rangle_X \quad \forall v \in X, \tag{1.1}$$

where $f \in X'$ is given and $\dot{u}$ stands for the generalized derivative of u.

The concept of a hemivariational inequality is introduced by Panagiotopoulos (for example, see [Naniewicz and Panagiotopoulos (1995)]). The background of these problems are in physics, especially in solid mechanics, where nonmonotone, multivalued constitutive laws lead to hemivariational inequalities. We refer to Carl (1996), Liu (1999), Liu and Simon (2001), Miettinen and Panagiotopoulos (1999), Migórski (2000) and references therein to see the applications of evolution hemivariational inequalities. More specific, Naniewicz and Panagiotopoulos (1995) dealt with the stationary **Problem P. For Problem P**, Carl (1996) used the method of upper and lower solutions and Miettinen and Panagiotopoulos (1999) used a regularized approximating method, both of them got the existence results of **Problem P** with A being a quasilinear elliptic operator of second order, while Liu (1999), Liu and Simon (2001) and Migórski (2000) studied the existence of solution for **Problem P** using the theory of pseudomonotone operators, if the operators involved satisfy certain coerciveness conditions.

However, engineering, economic and stochastic models lead to **Problem P** with non-coercive operators. For both mathematics and applications, in the present paper, we do not assume that the operators involved satisfy any sorts of coerciveness conditions. The existence theorems of the **Problem P**, available in the literature, become inefficient in this situation.

In order to handle the present situation, we intend to employ the so-called Browder–Tikhonov regularization method. Using this method, Giannessi and Khan (2000), Isac (1993) dealt with quasi variational inequalities and complementarity problems, respectively.

We assume that instead of the exact data (A, G, f) only the noisy data $(A_{\alpha_n}, G_{\beta_n}, f_{\gamma_n})$ are available. Here $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$, are sequences of positive reals. Under the relationship between exact data and noisy data, we shall show that sequences of solutions for the approximating **Problem P** converge to the solutions of the original **Problem P**.

2. Preliminaries

For any Banach space B, let $h\colon B \to R$ be Lipschitz near a given point $u \in B$, and let v be any other vector in Banach space B. The generalized directional derivative of h at u in the direction v, denoted by $h^0(u,v)$, is defined as follows:

$$h^0(u,v) = \limsup_{w \to u, \lambda \to 0^+} \lambda^{-1}[h(w+\lambda v) - h(w)],$$

where, of course, $w \in B$ and λ is a positive scalar (cf. [Clarke (1983)]), by means of which the Clarke's generalized gradient of h at u, denoted by $\partial h(u)$, is the subset of B' (the dual space of B) given by

$$\partial h(u) = \{w \in B' \colon h^0(u,v) \geqslant \langle w, v\rangle_B \text{ for all } v \in B\}. \tag{2.1}$$

Let $g(t,u)$: $I\times H\to R$ is measurable in $t\in I$, locally Lipschitz in H and $g(t,0)=0$ for a.e. $t\in I$. As in [Liu (1999)], we impose upon g the following growth condition:

$$\|w\|_H\leqslant c(1+\|v\|_H)\quad \text{for } w\in\partial g(t,v),\ t\in I,\ v\in H. \tag{2.2}$$

for a positive constant c independent of $t\in I$ and $v\in H$.

It follows from [Liu (1999)] that the functional G: $L^2(I,H)\to R$ of the type

$$G(u)=\int_I g(t,u(t))\mathrm{d}t,\quad \text{for any } u\in L^2(I,H), \tag{2.3}$$

is well-defined and Lipschitz continuous on the bounded subsets of $L^2(I,H)$. The generalized gradient $\partial G(u)$ of G at u on $L^2(I,H)$ is the subset of $L^2(I,H)$ given by

$$\partial(G|_{L^2})(u)=\{w\in L^2(I,H): G^0(u,v)\geqslant\langle w,v\rangle_{L^2}\quad \text{for all } v\in L^2(I,H)\}$$

where $G^0(u,v)=\limsup_{w\to u,\lambda\to 0_+}\lambda^{-1}[G(w+\lambda v)-G(w)]$.

Since $X(=L^2(I,V))$ is dense in $L^2(I,H)$, by means of [Clarke (1983)] (see also [Liu (1999)]), we have

$$\partial(G|_X)(u)=\partial(G|_{L^2})(u)\quad \forall u\in X. \tag{2.4}$$

Define L: $D(L)\subset X\to X'$, by $Lu=\dot{u}$, where $D(L)=\{v\in X|\dot{v}\in X', v(0)=v(T)\}$, $\dot{u}$ stands for the generalized derivative of u, i.e., $\int_I\dot{u}(t)\phi(t)\mathrm{d}t=-\int_I u(t)\dot{\phi}(t)\mathrm{d}t$ for all $\phi\in C_0^\infty(I)$. Then $\langle Lu,v\rangle_X=\int_I\langle\dot{u}(t),v(t)\rangle_V\mathrm{d}t$ for any $u\in D(L)$ and $v\in X$.

Since L is a closed densely linear maximal monotone map (see Zeidler, 1990, pp. 855 and pp. 897), therefore the graph of L is a closed set in $X\times X'$. So $Y\equiv D(L)$ equipped with the graph norm

$$\|u\|_Y=\|u\|_X+\|Lu\|_{X'},\quad \text{for any } u\in Y,$$

becomes a real reflexive Banach space.

Let us recall that a multivalued mapping M: $D(M)\subseteq X\to 2^{X'}$ is said to have the (S_+) property with respect to $D(L)$ if

(a) The set $M(u)$ is nonempty, bounded, closed and convex for each $u\in D(M)$;
(b) M is finitely weakly upper-semicontinuous, i.e., for each finite dimensional subspace E of X, M is an upper-semicontinuous mapping of $D(M)\cap E$ into X' supplied with the weak topology;
(c) For any sequence $\{u_n\}$ in $D(L)\cap D(M)$ converging weakly to an element u of $D(M), Lu_n$ converging weakly to Lu in X' and for any sequence $\{w_n\}$ in X' with $w_n\in M(u_n)$ for each $n\geqslant 1$, the condition $\limsup_{n\to\infty}\langle w_n,u_n-u\rangle_X\leqslant 0$ implies the strong convergence of $\{u_n\}$ to u in X and there exists a subsequence $\{w_{n_j}\}$ of $\{w_n\}$ such that $\{w_{n_j}\}$ converges weakly to $w\in M(u)$ in X'.

It is well-known that the conditions

$$\|J(u)\|_{X'} = \|u\|_X, \langle J(u), u\rangle_X = \|u\|_X^2, \quad \forall u \in X$$

determine a unique map J from X to X', which is called the duality map. In our case it is bijective bicontinuous strictly monotone and of class (S_+). For more details we refer to [Zeidler (1990)].

DEFINITION 1. Let m be a positive constant. By $\mathbf{A}(m)$ we denote the class of mappings A: $X \to X'$, which is bounded, demicontinuous and satisfies the following strongly monotone conditions:

$$\langle Au - Av, u - v\rangle_X \geqslant m\|u - v\|_X^2, \quad \forall u, v \in X. \tag{2.5}$$

By $\mathbf{G}(m)$ we denote the class of locally Lipschitz functions G: $X (\subseteq L^2(I, H)) \to R$ defined by (2.3), which are supposed to fulfill the inequality (2.2) and the following conditions of relaxed monotonicity:

$$\langle u^* - v^*, u - v\rangle_X \geqslant -m\|u - v\|_X^2, \quad \forall u, v \in X, \tag{2.6}$$

for any $u^* \in \partial(G|_X)(u)$ and $v^* \in \partial(G|_X)(v)$.

Remark. If $A \in \mathbf{A}(m)$ and $G \in \mathbf{G}(m)$, then we obviously have that $A + \partial G$ is monotone. But $A + \partial G$ may not be coercive in general.

Let $\{\epsilon_n\}, \epsilon_n > 0, (n = 1, 2, \ldots)$ be a sequence of positive reals which is (strictly) decreasing and converging to zero.

The relationship between the exact data and the noisy data is given through the following assumptions:

ASSUMPTION 1. There exists a continuous function τ_1: $R_+ \to R_+$ such that

$$\|A(u) - A_{\alpha_n}(u)\|_{X'} \leqslant \alpha_n \tau_1(\|u\|_X), \quad \forall u \in X.$$

ASSUMPTION 2. There exists a continuous function τ_2: $R_+ \to R_+$ such that

$$H(\partial(G|_X)(u), \partial(G_{\beta_n}|_X)(u)) \leqslant \beta_n \tau_2(\|u\|_X), \quad \forall u \in X,$$

where $H(Q, S) = \max\{\sup_{x \in Q} d(x, S), \sup_{y \in S} d(Q, y)\}$ is the Hausdorff distance between the sets Q and S.

ASSUMPTION 3. For $f_{\gamma_n} \in X'$, we have $\|f - f_{\gamma_n}\|_{X'} \leqslant \gamma_n$.

ASSUMPTION 4. For $n \to \infty$

$$\alpha_n, \beta_n, \gamma_n, \frac{\alpha_n}{\epsilon_n}, \frac{\beta_n}{\epsilon_n}, \frac{\gamma_n}{\epsilon_n} \to 0.$$

Consider the following Regularized Evolution Hemivariational Inequality: find $u_n \in Y$ such that

$$\langle \dot{u}_n, v - u_n\rangle_X + \langle A_{\alpha_n}(u_n) + \epsilon_n J(u_n) - f_{\gamma_n}, v - u_n\rangle_X + \\ + G^0_{\beta_n}(u_n, v - u_n) \geqslant 0, \quad \forall v \in X. \tag{2.7}$$

In the above, the operator $J: X \to X'$ is the regularizing operator, ϵ_n the regularization parameter and u_n is the regularized solution to the **Problem P**. Here the symbol $(\alpha_n, \beta_n, \gamma_n, \epsilon_n)$ shows the influence of the error parameters $\alpha_n, \beta_n, \gamma_n$ and the regularization parameters ϵ_n.

3. Main Results

In the first part of this section, we consider the relationship between solutions of the following inclusions in X' with generalized gradients

$$f \in Lu + Au + \partial(G|_X)(u), \tag{3.1}$$

and the regularized inclusions

$$f_{\gamma_n} \in Lu + A_{\alpha_n} u + \epsilon_n J(u) + \partial(G_{\beta_n}|_X)(u), \tag{3.2}$$

where $A, A_{\alpha_n} \in \mathbf{A}(m)$ and $G, G_{\beta_n} \in \mathbf{G}(m)$.

LEMMA 1. *All solutions of the inclusion* (3.1) *are solutions of the evolution hemivariational inequality* (1.1); *while all solutions of Regularized inclusion* (3.2) *are also solutions of Regularized evolution hemivariational inequality* (2.7).

Proof. Let u be a solution of (3.1). Then $u \in D(L)$. So $u \in X, u(0) = u(T)$ and there is $w \in \partial(G|_X)(u)$ such that

$$Lu + Au + w = f.$$

Scalar multiplying the above equality by $v \in X$, we have

$$\langle \dot{u}, v \rangle_X + \langle Au, v \rangle_X + \langle w, v \rangle_X = \langle f, v \rangle_X \quad \forall v \in X.$$

Since the generalized Clarke's gradient $\partial(G|_X)(u)$ is the subset of X given by [Clarke (1983)]

$$\partial(G|_X)(u) = \{w \in X' : G^0(u, v) \geqslant \langle w, v \rangle_X \quad \forall v \in X'\},$$

we obtain

$$\langle \dot{u}, v \rangle_X + \langle Au, v \rangle_X + G^0(u, v) \geqslant \langle f, v \rangle_X \quad \forall v \in X,$$

i.e., u is also a solution of the evolution hemivariational inequality (1.1).

Similarly, we can show that the second part of the Theorem is true. This completes the proof of the Theorem.

THEOREM 1. *Assume that the mappings* $A_{\alpha_n} \in \mathbf{A}(m)$ *and* $G_{\beta_n} \in \mathbf{G}(m)$. *Then, for each* $n \in \mathcal{N}$ *and given* $f_{\gamma_n} \in X'$ *there exists at least one solution of the inclusion* (3.2).

Proof. For each $\epsilon_n > 0$, we define $S_n: X \to 2^{X'}$ by

$$S_n = A_{\alpha_n} + \epsilon_n J + \partial(G_{\beta_n}|_X).$$

At first, we shall show that $S_n: X \to 2^{X'}$ has the (S_+) property with respect to $D(L)$.

Similar to the proof of Lemma 3.2 in [Liu (1999)], we easily prove that the conditions (a) and (b) in the definition of (S_+) mappings hold. Now let $\{u_k\}$ in Y with $u_k \to u$ weakly in $X, Lu_k \to Lu$ weakly in X', and $w_k \in \partial(G_{\beta_n}|_X)(u_k)$ such that

$$\limsup_{k\to\infty} \langle A_{\alpha_n}(u_k) + \epsilon_n J(u_k) + w_k, u_k - u\rangle_X \leqslant 0.$$

From the weak convergence of $\{u_n\}$ in Y, for $\forall w \in \partial(G_{\beta_n}|_X)(u)$ we obtain

$$\limsup_{k\to\infty} \langle A_{\alpha_n}(u_k) - A_{\alpha_n}(u) + w_k - w + \epsilon_n J(u_k), u_k - u\rangle_X \leqslant 0.$$

Since $A_{\alpha_n} + \partial(G_{\beta_n}|_X)$ is monotone by the Remark above, we conclude that

$$\limsup_{k\to\infty} \langle J(u_k), u_k - u\rangle_X \leqslant 0.$$

Because the duality map J is bijective bicontinuous strictly monotone and of class (S_+), we obtain from the above inequality

$$u_k \to u \text{ in } X, \quad \text{as } k \to \infty, \tag{3.3}$$

$$J(u_k) \to J(u) \text{ in } X', \quad \text{as } k \to \infty. \tag{3.4}$$

Using the demicontinuity of A_{α_n} and the weak*-closedness of the generalized gradient $\partial(G_{\beta_n}|_X)$ (see Clarke, 1983, pp. 29), we have from (3.3)

$$A_{\alpha_n}(u_k) \rightharpoonup A_{\alpha_n}(u) \text{ in } X', \quad \text{as } k \to \infty, \tag{3.5}$$

$$w_k \rightharpoonup w \in \partial(G_{\beta_n}|_X)(u) \text{ in } X', \quad \text{as } k \to \infty. \tag{3.6}$$

Combining (3.4)–(3.6), we get that there exists $w \in \partial(G_{\beta_n}|_X)(u)$ such that

$$A_{\alpha_n}(u_k) + \epsilon_n J(u_k) + w_k \rightharpoonup_{k\to\infty} A_{\alpha_n}(u) + \epsilon_n J(u) + w \in A_{\alpha_n}(u) + \\ + \epsilon_n J(u) + \partial(G_{\beta_n}|_X)(u).$$

Therefore, S_n has the (S_+) property.

In the following, we shall show that the operator S_n is coercive. $\forall w \in \partial(G_{\beta_n}|_X)(u), u \in X$, and $\forall w_0 \in \partial(G_{\beta_n}|_X)(0)$ we have from (2.5), (2.6)

$$\langle A_{\alpha_n}(u), u\rangle_X \geqslant m\|u\|_X^2 - \|A_{\alpha_n}(0)\|_{X'}\|u\|_X \tag{3.7}$$

$$\langle w, u\rangle_X \geqslant -m\|u\|_X^2 - \|w_0\|_{X'}\|u\|_X \tag{3.8}$$

Combining (3.7), (3.8), yields

$$\langle A_{\alpha_n} u + \epsilon_n J(u) + w, u\rangle_X \geqslant \epsilon_n \|u\|_X^2 - (\|A_{\alpha_n}(0)\|_{X'} + \|w_0\|_X)\|u\|_X.$$

From the fact that the above estimate is valid for $\forall w \in \partial(G_{\beta_n}|_X)(u)$, we deduce that

$$\inf\{\langle u^*, u\rangle_X : u^* \in S_n(u)\}/\|u\|_X \to \infty \quad \text{as } \|u\|_X \to \infty,$$

which proves the coercivity of S_n for all $\epsilon_n > 0$.

From the well-known surjectivity result for multi-valued (S_+) mappings with coerciveness (for example, see [Liu (1999)]), for each $f_{\gamma_n} \in X'$, there exists $u_n \in X$ such that

$$f_{\gamma_n} \in Lu_n + A_{\alpha_n}(u_n) + \epsilon_n J(u_n) + \partial(G_{\beta_n}|_X)(u_n).$$

This completes the proof.

Furthermore, we shall prove

THEOREM 2. *Assume that the mappings* $A, A_{\alpha_n} \in \mathbf{A}(m)$ *and* $G, G_{\beta_n} \in \mathbf{G}(m)$ *and Assumptions* 1–4 *hold. If the inclusion* (3.1) *is solvable, then, the sequence* $\{u_n\}$ *of solutions for the regularized inclusions* (3.2) *is uniformly bounded in* Y.

Proof. By Theorem 1, we may assume that every u_n satisfies the following regularized inclusion:

$$f_{\gamma_n} \in L_{u_n} + A_{\alpha_n}(u_n) + \epsilon_n J(u_n) + \partial(G_{\beta_n}|_X)(u_n), \quad u_n \in D(L). \tag{3.9}$$

From the assumption that the inclusion (3.1) is solvable, it is clear that there exists at least one $u \in Y$ such that

$$-f \in -Lu - Au - \partial(G|_X)(u), \tag{3.10}$$

Summing-up the above two equalities side by side, we obtain

$$f_{\gamma_n} - f \in Lu_n - Lu + A_{\alpha_n}(u_n) - A(u) + \epsilon_n J(u_n) + \partial(G_{\beta_n}|_X)(u_n) - \partial(G|_X)(u),$$

which implies that there exist $w_n \in \partial(G_{\beta_n}|_X)(u_n)$ and $w \in \partial(G|_X)(u)$ such that

$$\begin{aligned}&\langle Lu_n - Lu, u_n - u\rangle_X + \langle A_{\alpha_n}(u_n) - A(u), u_n - u\rangle_X + \\ &\quad + \epsilon_n \langle J(u_n), u_n - u\rangle_X + \langle w_n - w, u_n - u\rangle_X \\ &= \langle f_{\gamma_n} - f, u_n - u\rangle_X\end{aligned}$$

Therefore, we have for any $\bar{w}_n \in \partial(G_{\beta_n}|_X)(u)$

$$\begin{aligned}&\langle f_{\gamma_n} - f, u_n - u\rangle_X - \langle Lu_n - Lu, u_n - u\rangle_X - \langle A\alpha_n(u) - A(u), u_n - u\rangle_X - \\ &\quad - \langle (A_{\alpha_n}(u_n) + w_n) - (A\alpha_n(u) + \bar{w}_n), u_n - u\rangle_X + \langle \bar{w}_n - w, u - u_n\rangle_X \\ &= \epsilon_n \langle J(u_n), u_n - u\rangle_X\end{aligned}$$

By the monotonicity of $L, A_{\alpha_n} + \partial(G_{\beta_n}|_X)$ and Assumptions 1–2, we get

$$\begin{aligned}&\gamma_n \|u_n - u\|_X + \alpha_n \tau_1(\|u\|_X)\|u - u_n\|_X + \|\bar{w}_n - w\|_X \|u - u_n\|_X + \\ &\quad + \epsilon_n \|u_n\|_X \|u\|_X \geqslant \epsilon_n \|u_n\|_X^2.\end{aligned} \tag{3.11}$$

Since for any $\bar{w}_n \in \partial(G_{\beta_n}|_X)(u)$ the above inequality holds, we can choose that $\bar{w}_n \in \partial(G_{\beta_n}|_X)(u)$ such that

$$\begin{aligned}\|\bar{w}_n - w\|_X &\leqslant d(\partial(G_{\beta_n}|_X)(u), w) + \epsilon_n \\ &\leqslant \sup_{w \in \partial(G|_X)(u)} d(\partial(G_{\beta_n}|_X)(u), w) + \epsilon_n \\ &\leqslant H(\partial(G_{\beta_n}|_X)(u), \partial(G|_X)(u)) + \epsilon_n \\ &\leqslant \beta_n \tau_2(\|u\|_X) + \epsilon_n\end{aligned} \tag{3.12}$$

By means of this inequality, the inequality (3.11) can be expressed as

$$\gamma_n\|u_n-u\|_X+\alpha_n\tau_1(\|u\|_X)\|u-u_n\|_X+\beta_n\tau_2(\|u\|_X)\|u-u_n\|_X+ \\ +\epsilon_n\|u-u_n\|_X+\epsilon_n\|u_n\|_X\|u\|_X\geqslant\epsilon_n\|u_n\|_X^2$$

Therefore, we obtain

$$\frac{\gamma_n}{\epsilon_n}\|u_n-u\|_X+\frac{\alpha_n}{\epsilon_n}\tau_1(\|u\|_X)\|u-u_n\|_X+\frac{\beta_n}{\epsilon_n}\tau_2(\|u\|_X)\|u-u_n\|_X+ \\ +(\|u-u_n\|_X)+\|u_n\|_X\|u\|_X \\ \geqslant\|u_n\|_X^2.$$

The above inequality, in view of Assumption 4, confirms the existence of a constant C, such that

$$\|u_n\|_X\leqslant C,\quad\forall n\in\mathcal{N} \tag{3.13}$$

In order to show that the whole sequence $\{u_n\}$ is bounded in Y, we still have to prove that $\|Lu_n\|_{X'}\leqslant Const$. In terms of (3.9), there exists $w_n\in\partial(G_{\beta_n}|_X)(u_n)$ such that

$$f_{\gamma_n}=Lu_n+A_{\alpha_n}(u_n)+\epsilon_nJ(u_n)+w_n,$$

which implies that for $\forall\bar{w}_n\in\partial(G|_X)(u_n)$,

$$\begin{aligned}\|Lu_n\|_{X'}&\leqslant\|A_{\alpha_n}(u_n)-A(u_n)\|_{X'}+\|A(u_n)\|_{X'}+\epsilon_n\|u_n\|_X+\\&\quad+\|f_{\gamma_n}-f\|_{X'}+\|f\|_{X'}+\|w_n-\bar{w}_n\|_{X'}+\|\bar{w}_n\|_{X'}\\&\leqslant\alpha_n\tau_n(\|u_n\|_X)+\|A(u_n)\|_{X'}+\epsilon_n\|u_n\|_X+\\&\quad+\gamma_n+\|f\|_{X'}+\|w_n-\bar{w}_n\|_{X'}+\|\bar{w}_n\|_{X'}.\end{aligned} \tag{3.14}$$

Like (3.12), we can choose $\bar{w}_n\in\partial(G|_X)(u_n)$ such that

$$\begin{aligned}\|w_n-\bar{w}_n\|_{X'}&\leqslant d(w_n,\partial(G|_X)(u_n))+\epsilon_n\\&\leqslant\sup_{w_n\in\partial(G_{\beta_n}|_X)(u_n)}d(w_n,\partial(G|_X)(u_n))+\epsilon_n\\&\leqslant H(\partial(G_{\beta_n}|_X)(u_n),\partial(G|_X)(u_n))+\epsilon_n\\&\leqslant\beta_n\tau_2(\|u_n\|_X)+\epsilon_n.\end{aligned} \tag{3.15}$$

On substituting the above estimate to (3.14), we have

$$\begin{aligned}\|Lu_n\|_{X'}&\leqslant\alpha_n\tau_1(\|u_n\|_X)+\|A(u_n)\|_{X'}+\epsilon_n\|u_n\|_X+\\&\quad+\gamma_n+\|f\|_{X'}+\beta_n\tau_2(\|u_n\|_X)+\epsilon_n+\|\bar{w}_n\|_{X'}.\end{aligned}$$

Since the operator A and multivalued operator $\partial(G|_X)$ are bounded, by means of Assumption 4 and (3.13), we reach the conclusion that

$$\|Lu_n\|_{X'}\leqslant\text{Const}.$$

The proof is complete.

THEOREM 3. *Assume that the hypotheses of Theorem 2 hold. Then, every weak limit point of a subsequence of* $\{u_n\}$, *still denoted by* $\{u_n\}$, *is a solution to* **Problem P**. *Furthermore, if* **Problem P** *is uniquely solvable, then the whole sequence* $\{u_n\}$ *converges to the solution.*

Proof. Since the sequence $\{u_n\}$ is uniformly bounded in Y from Theorem 2, it is weakly compact by the reflexivity of the space Y. Therefore, it is always possible to extract a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, such that

$$u_n \rightharpoonup u \in X, Lu_n \rightharpoonup Lu \quad \text{as } n \to \infty. \tag{3.16}$$

As u_n is a solution to the regularized inclusion (3.2), so there exists $\bar{w}_n \in \partial(G_{\beta_n}|_X)(u_n)$ such that

$$f_{\gamma_n} = Lu_n + A_{\alpha_n}(u_n) + \epsilon_n J(u_n) + \bar{w}_n. \tag{3.17}$$

Similar to (3.15), we first choose $w_n \in \partial(G|_X)(u_n)$ such that

$$\|w_n - \bar{w}_n\|_{X'} \leqslant \beta_n \tau_2(\|u_n\|_X) + \epsilon_n. \tag{3.18}$$

Therefore, we have

$$\begin{aligned}\langle A(u_n) + w_n, u_n - u\rangle_X = {} & \langle f_{\gamma_n} - f, u_n - u\rangle_X - \langle Lu_n - Lu, u_n - u\rangle_X + \\ & + \langle f, u_n - u\rangle_X - \langle Lu, u_n - u\rangle_X + \\ & + \langle A(u_n) - A_{\alpha_n}(u_n), u_n - u\rangle_X - \epsilon_n \langle J(u_n), u_n - u\rangle_X + \\ & + \langle w_n - \bar{w}_n, u_n - u\rangle_X.\end{aligned}$$

Using the weak convergence of $\{u_n\}$ in Y, the assumptions 1–4 and (3.18), we deduce

$$\limsup_{n\to\infty} \langle A(u_n) + w_n, u_n - u\rangle_X \leqslant 0,$$

which implies

$$\limsup_{n\to\infty} \langle A(u_n), u_n - u\rangle_X + \limsup_{n\to\infty} \langle w_n, u_n - u\rangle_X \leqslant 0 \tag{3.19}$$

From (2.4), we have

$$w_n \in \partial(G|_X)(u_n) = \partial(G|_{L^2})(u_n)$$

which implies $w_n \in L^2(I, H)$. Hence

$$\langle w_n, u_n - u\rangle_X = \langle w_n, u_n - u\rangle_{L^2}.$$

The compactness of the embedding $V \subseteq H$ implies that Y is compactly imbedded into $L^2(I, H)$ (see Zeidler, 1990, pp. 450). Therefore, the weak convergence of $\{u_n\}$ in Y implies the strong convergence of $\{u_n\}$ in $L^2(I, H)$. Furthermore, by Lemma 3.1 in [Liu (1999)], one deduces that $\{w_k\}$ is a bounded sequence in $L^2(I, H)$. By the basic properties of the generalized gradient [Clarke (1983)], passing to a subsequence if necessary, we may assume

$$w_n \rightharpoonup w \in \partial(G|_{L^2})(u) (= \partial(G|_X(u)) \text{ in } L^2(I, H). \tag{3.20}$$

Therefore, we have

$$\liminf \langle w_n, u_n - u \rangle_X = \liminf \langle w_n, u_n - u \rangle_{L^2} = 0 \tag{3.21}$$

By the weak convergence of $\{u_n\}$ in Y, (3.19) and (3.21) we have

$$\limsup_{n\to\infty} \langle A(u_n) - A(u), u_n - u \rangle_X \leqslant 0$$

In virtue of (2.5), we obtain

$$u_n \to u \quad \text{in } X. \tag{3.22}$$

Using the demicontinuity of A and the weak*-closedness of the generalized gradient $\partial(G|_X)$ (see Clarke, 1983, pp. 29), we have from (3.22)

$$A(u_n) \rightharpoonup A(u) \text{ in } X', \quad \text{as } n \to \infty.$$

$$w_n \rightharpoonup w \in \partial(G|_X)(u) \text{ in } X', \quad \text{as } n \to \infty.$$

By the assumptions 1–4 and (3.18) we also have

$$f_{\gamma_n} \to f \text{ in } X', \quad \text{as } n \to \infty; \tag{3.23}$$

$$\epsilon_n J(u_n) \to 0 \text{ in } X', \quad \text{as } n \to \infty; \tag{3.24}$$

$$A_{\alpha_n}(u_n) - A(u_n) \to 0 \text{ in } X', \quad \text{as } n \to \infty;$$

$$w_n - \bar{w}_n \to 0 \text{ in } X', \quad \text{as } n \to \infty.$$

Therefore, we obtain

$$A_{\alpha_n}(u_n) \rightharpoonup A(u) \text{ in } X', \quad \text{as } n \to \infty, \tag{3.25}$$

$$\bar{w}_n \rightharpoonup w \in \partial(G|_X)(u) \text{ in } X', \quad \text{as } n \to \infty. \tag{3.26}$$

From (3.16), (3.23)–(3.26), let $n \to \infty$ in (3.17). We conclude

$$f = Lu + A(u) + w \in Lu + A(u) + \partial(G|_X)(u).$$

This together with Lemma 1 implies that u is a solution of **Problem P**. Furthermore if **Problem P** is uniquely solvable, then clearly u is the unique limit of any weakly convergent subsequence of $\{u_n\}$. Therefore we have the convergence of the whole sequence $\{u_n\}$ to u. The proof is complete.

Acknowledgements

The author thanks the referees for comments and suggestions on the original manuscript which led to its improvement.

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